Homework 5 Solutions

Math 131A-3

1. Problems from Ross.

(17.3) (a) By assumption, $\cos(x)$ is continuous, so since products of continuous functions are continuous, $\cos^4 x$ is continuous. Moreover, constant funtions are continuous, and sums of continuous functions are continuous, so $1 + \cos^4 x$ is continuous. Finally, $\log x$ is continuous on its domain, all positive numbers, so given that $1 + \cos^4 x > 1$ and compositions of continuous functions are continuous, $\log(1 + \cos^4 x)$ is continuous on $\mathbb{R}$. (b) Because 2^x and x^2 are continuous functions and compositions of continuous functions are continuous, 2^{x^2} is also continuous.

(17.10) In each case, it suffices to find a sequence $s_n \to x_0$ such that $\lim f(s_n) \neq f(x_0)$.

(a) Let $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Consider the sequence (s_n) where $s_n = \frac{1}{n}$. Then $\lim s_n = 0$, but $\lim f(s_n) = \lim f(\frac{1}{n}) = 1 \neq 0$. So f is not continuous at 0.

(b) Let $g(x) = \sin(\frac{1}{x})$ for $x \neq 0$ and $g(0) = 0$. Consider the sequence (s_n) where $s_n = \frac{1}{(2n+\frac{1}{2})\pi}$. Then $\lim s_n = 0$, but $\lim f(s_n) = \lim \sin((2n + \frac{1}{2})\pi) = \lim 1 = 1 \neq f(0)$. Ergo f is not continuous at 0.

(c)Let $\operatorname{sgn}(x) = -1$ for $x < 0$, $\operatorname{sgn}(x) = 1 for x > 0$, and $\operatorname{sgn}(0) = 0$. Again consider the sequence (s_n) such that $s_n = \frac{1}{n}$. Then $\lim s_n = 0$, but $\lim f(s_n) = \lim 1 = 1$. So f is not continuous at 0.

(17.12) (a) Suppose f is a continuous real-valued function on (a, b) such that $f(r) = 0$ for all r rational in (a, b). Let $x \in (a, b)$. For each $n \in \mathbb{N}$, choose a rational number r_n in the interval $(x - \frac{1}{n}, x) \cap (a, b)$. (We know this is possible because every interval contains a rational number.) Then $|x - r_n| < \frac{1}{n}$, so $\lim r_n = x$. Therefore by continuity, $f(x) = \lim f(r_n) = \lim 0 = 0$.

(b) Consider the function $f - g(x)$ on (a, b), which is continuous on (a, b) since both f and g are continuous on $a, b)$. Since $f(r) = g(r)$ on all rational $r \in (a, b)$, $f - g(r) = 0$ on all rational $r \in (a, b)$. Ergo by part (a), $f - g(x) = 0$ for all $x \in (a, b)$, so $f(x) = g(x)$ for all $x \in (a, b)$.

Ergo continuous functions on intervals are determined by their values on the rational numbers!

(18.4)Let $S \subset \mathbb{R}$. Suppose there exists a sequence (x_n) in S such that $x_n \to x_0 \notin S$. Let $f(x) = \frac{1}{x - x_0}$. Then f is well-defined on S since $x_0 \notin S$, and is continuous since it is a quotient of continuous functions such that the denominator is nonzero. Now for any $M > 0$, choose N such that $n > N$ implies $|x_n - x_0| < \frac{1}{M}$. Then for $n > N$, $|f(x_n)| = \frac{1}{|x_n - x_0|} > M$. Since M was arbitrary, f is unbounded on S.

So any set which is not closed (i.e. does not contain all its limit points) is the domain of some unbounded continuous function.

(18.7) Let $f(x) = xe^x$. Since products of continuous functions are continuous, f is continuous on $\mathbb{R}$. Observe that $f(0) = 0$ and $f(1) = e \approx 2.718$. Since $f(0) < 2 < f(1)$, by the Intermediate Value Theorem, there is some x in $(0, 1)$ such that $f(x) = 2$.

(18.10) Let f be a continuous function on $[0, 2]$ such that $f(0) = f(2)$. Consider the function $g(x) = f(x+1) - f(x)$ on $[0, 1]$. Observe that $f(x+1)$ is a composition of the continuous functions $f(x)$ and $x+1$, hence continuous, so g is a difference of continuous functions and therefore continuous. Moreover, $g(0) = f(1) - f(0) = f(1) - f(2) = -[f(2) - f(1)] = -g(1)$. Since $g(0) = -g(1)$, either $g(0) \leq 0 \leq g(1)$ or $g(0) \geq 0 \geq g(1)$; in either case, by the Intermediate Value Theorem, there exists $x \in [0, 1]$ such that $g(x) = 0$. This implies that $0 = f(x+1) - f(x)$, or equivalently $f(x+1) = f(x)$. Let $y = x + 1$, then x, y have the property that $|y - x| = 1$ and $f(x) = f(y)$.

Ergo if you start a car, drive for two hours, and then stop, at some point during the second hour you will be driving exactly the speed you were driving an hour ago.

(19.1) (a) $f(x) = x^{17} \sin x - e^x \cos(3x)$ is built by sums, products, and compositions of continuous functions, hence is continuous. Ergo since $[0, \pi]$ is a closed interval, by Theorem 19.2 f is uniformly continuous on $[0, \pi]$.

(c) $f(x) = x^3$ can be extended continuously from $(0, 1)$ to $[0, 1]$ by letting $f(0) = 0$ and $f(1) = 1$. Ergo by Theorem 19.5, f is uniformly continuous on $(0, 1)$.

(f) If $f(x) = \sin(\frac{1}{x^2})$ on $(0, 1]$, consider the Cauchy sequence (s_n) where $s_n = \frac{1}{\sqrt{\frac{n\pi}{2}}}$ for $n \geq 1$. Then $f(s_n) = \sin(\frac{n\pi}{2})$, so the sequence $(f(s_n))$ is $(1, 0, -1, 0, 1, 0, \cdots)$, which is not Cauchy. Since uniformly continuous functions map Cauchy sequences to Cauchy sequences by Theorem 19.4, f is not uniformly continuous on $(0, 1]$.

(g)Let $f(x) = x^2 \sin \frac{1}{x}$ on $(0, 1]$. We claim that f may be extended to $\tilde{f}$ continuous on $[0, 1]$ by setting $\tilde{f}(0) = 0$. For if (x_n) is any sequence in $(0, 1]$ converging to 0, then $0 \leq |f(x_n)| = |x_n^2 \sin \frac{1}{x}| \leq |x_n^2|$, so since $x_n^2 \to 0$, $\tilde{f}(x_n) \to 0 = \tilde{f}(0)$. Hence $\tilde{f}$ is continuous at 0, and by Theorem 19.5 the existence of a continuous extension to $[0, 1]$

suffices to show that f is uniformly continuous on $(0, 1]$.

(19.2)(b) Let $f(x) = x^2$ on $[0, 3]$. Let $\epsilon > 0$, and set $\delta = \frac{\epsilon}{6}$. Then for $x, y \in [0, 3]$, if $|x - y| < \delta$, $|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| < \frac{\epsilon}{6}(6) = \epsilon$. Ergo f is uniformly continuous on $[0, 3]$.

(19.4)(a) Let f be uniformly continuous on a bounded set S. Suppose that f is unbounded on S. Then for any $N \in \mathbb{N}$, there is an $x_n \in S$ such that $|f(x_n)| > N$. Consider the sequence (x_n). By the Bolzano-Weierstrass Theorem, some subsequence (x_{n_k}) converges, hence is Cauchy. But by Theorem 19.4, since f is uniformly continuous, f maps Cauchy sequences to Cauchy sequences, so $(f(x_{n_k}))$ is a Cauchy sequence, hence converges to some real number. However, by construction $\lim f(x_{n_k}) = \infty$. This is a contradiction, so f must be bounded on S.

(b) Observe that $f(x) = \frac{1}{x^2}$ is not bounded on the bounded set $(0, 1)$, so f cannot be uniformly continuous on $(0, 1)$.

2. The stars over Babylon function.

- For any rational number r, suppose for the sake of contradiction that $r + \sqrt{2}$ is rational. Then since the rationals are closed under taking additive inverses and addition, we sould have $(r + \sqrt{2}) + -(r)$ rational, but this implies that $\sqrt{2}$ is rational, which we know to be false. Ergo $r + \sqrt{2}$ is always irrational. Now given any interval (a, b), we know the interval $(a - \sqrt{2}, b - \sqrt{2})$ contains a rational number r, so (a, b) contains a number of the form $r + \sqrt{2}$. Hence every interval contains an irrational number.

- Let $x_0 = \frac{p}{q}$ be rational, such that $f(x_0) = \frac{1}{q}$. Then for every $n \in \mathbb{N}$, choose an irrational $x_n \in (x_0 - \frac{1}{n}, x_0) \cap (0, 1]$. The sequence (x_n) has the property that $|x_0 - x_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$, so $x_n \to x_0$. However, since x_n is irrational, $f(x_n) = 0$ for all n, so $\lim f(x_n) = 0 \neq \frac{1}{q} = f(x_0)$. So f is discontinuous at x_0.

- Let x_0 be irrational, so that $f(x_0) = 0$. Observe that the set of values our function f takes is $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Notice that for every $n \in N$, if $r \in (0, 1]$ has $f(r) = \frac{1}{n}$, it must be the case that r can be written as $\frac{i}{n}$ for some i. Therefore, if we let

$$\delta_N = \min\{|x_0 - \frac{i}{n}| : 0 \leq i \leq n \leq N,\ i, n \in \mathbb{N}\},$$

we see that for any $n \leq N$, $(x_0 - \delta_N, x_0 + \delta_N)$ contains no r such that $f(r) = \frac{1}{n}$. Ergo $|x - x_0| < \delta_N$ implies that $|f(x) - f(x_0)| < \frac{1}{N}$. Hence f is continuous at x_0.